

# Constructing circuit codes by permuting initial sequences

Ed Wynn

**Abstract**—Two new constructions are presented for coils and snakes in the hypercube. Improvements are made on the best known results for snake-in-the-box coils of dimensions 9, 10 and 11, and for some other circuit codes of dimensions between 8 and 13. In the first construction, circuit codes are generated from permuted copies of an initial transition sequence; the multiple copies constrain the search, so that long codes can be found relatively efficiently. In the second construction, two lower-dimensional paths are joined together with only one or two changes in the highest dimension; this requires a search for a permutation of the second sequence to fit around the first. It is possible to investigate sequences of vertices of the hypercube, including circuit codes, by connecting the corresponding vertices in an extended graph related to the hypercube. As an example of this, invertible circuit codes are briefly discussed.

**Index Terms**—Binary sequence, circuit code, coil, hypercube, snake in the box.

## I. INTRODUCTION

Let  $I^d$  be the graph of the  $d$ -dimensional hypercube. That is, the vertex set of  $I^d$  is  $\{0, 1\}^d$ , and two vertices are connected by an edge if and only if they differ in exactly one coordinate. A  $d$ -dimensional *circuit code* of length  $N$  and spread  $k \geq 1$  is a simple circuit  $(x_0, x_1, \dots, x_{N-1}, x_0)$  in  $I^d$  with the property that  $D(x_i, x_j) \geq \min(k, j - i, N + i - j)$  for all

E-mail ed.wynn@zoho.com

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$0 \leq i < j \leq N - 1$  where  $D(x_i, x_j)$  is the number of coordinates of  $I^d$  in which  $x_i$  and  $x_j$  differ. In other words, if two nodes are part of the circuit and the distance between them is  $i \leq k$ , then they must be connected directly by  $i$  transitions in the sequence.

A  $d$ -dimensional circuit code of spread 2 is here called a *d-coil*, following the terminology of [1]. Discovering long *d-coils* is known as the snake-in-the-box problem [2], and *d-coils* have been called snakes [3]. In this work, a *d-snake* is a simple path in  $I^d$  with spread 2.

A circuit code can be described by its *transition sequence*  $(c_0, c_1, \dots, c_{N-1})$ , where  $c_i$  specifies the coordinate that changes from  $x_i$  to  $x_{i+1}$  (with wraparound modulo  $N$ ). This paper presents two new constructions of circuit codes and some results of these constructions.

## II. PERMUTED CIRCUIT CODES

From an *initial sequence*  $\mathbf{c}^{(0)} = (c_0^{(0)}, c_1^{(0)}, \dots, c_{L-1}^{(0)})$ , we define *permuted sequences*  $\mathbf{c}^{(p)}$  by  $c_i^{(p)} = \pi(c_i^{(p-1)})$  for  $p \geq 1$  and  $0 \leq i \leq L - 1$ , where  $\pi$  is a permutation of  $\{0, 1, \dots, d - 1\}$ . A *permuted circuit code* of period  $P$  is then constructed as a circuit code whose transition sequence is an initial sequence followed by  $P - 1$  permuted sequences:  $(\mathbf{c}^{(0)}, \mathbf{c}^{(1)}, \dots, \mathbf{c}^{(P-1)})$ . It is convenient to divide the vertices into the corresponding sequences of length  $L$ : we define  $x_i^{(p)}$  to be  $x_{pL+i}$  for  $p \geq 0$  and  $0 \leq i \leq L - 1$ .

An example of a permuted circuit code is one of the four longest 6-coils, with  $N = 26$ . This has  $P = 2$ . The initial sequence is  $(0, 1, 2, 0, 3, 1, 4, 0, 2, 5, 3, 1, 2)$ , and the remainder is a simple permutation (merely swapping 2 and 4):  $(0, 1, 4, 0, 3, 1, 2, 0, 4, 5, 3, 1, 4)$ . (The other three 6-coils with  $N = 26$  can be described using terminology of later sections: one is asymmetric; one is natural; and one is invertible.)

When the initial vertex  $x_0$  is assumed without loss of generality to be 0, then the *initial leap* is defined to be the coordinate vector of  $x_0^{(1)}$ . This vector in  $I^d$  will involve a change or no change in each coordinate, according to whether the initial sequence has an even or odd number of changes in that coordinate. So, for the example of an initial sequence of in the previous paragraph, the pair of changes in coordinate 3 cancels out but all other coordinates have odd numbers of changes, so the initial leap is  $(1, 1, 1, 0, 1, 1)$ .

An algorithm for constructing permuted circuit codes is as follows:

- All permutations  $\pi$  are generated, up to conjugacy.
- For each permutation, each of the  $2^d$  possible vectors is proposed in turn as a possible initial leap. The initial leap is proposed before the initial sequence, or even the length of the initial sequence, is known.
- When a permutation and an initial leap have been proposed, then successive vertices  $x_0^{(1)}, x_0^{(2)}, \dots$  can then be deduced: coordinate  $i$  of one leap vector is equal to coordinate  $\pi(i)$  of the next. These vertices are generated until either the  $k$ -spread condition is violated (in which case the initial leap is rejected) or the initial vertex is revisited (so that  $x_0^{(P)} = x_0^{(0)}$  for some  $P$ ). In this way, the period  $P$  of a permuted circuit code can be deduced from its permutation and its initial leap. To return to

the example above, a permutation 01(24)35 and an initial leap  $(1, 1, 1, 0, 1, 1)$  would be considered. The permuted leap is the same as the initial leap, which returns to the initial vertex with  $P = 2$ .

- The *skeleton* of initial vertices  $(x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(P-1)}, x_0^{(0)})$  is compared to previous skeletons from the same permutation, and duplicates (up to isomorphism in  $I^d$ ) are rejected. Effectively, this is a test whether the proposed initial leap vector is equal to  $\sigma(\mathbf{v})$ , where  $\mathbf{v}$  is a leap vector that has already been proposed, and  $\sigma$  is a permutation that commutes with  $\pi$ . One method for conducting this test is described in Section V.
- For a suitable permutation and initial leap, an exhaustive search with backtracking is then conducted for initial sequences that link  $x_0^{(0)}$  to  $x_0^{(1)}$ . Whenever a new change in  $c_i^{(0)}$  is proposed, a new vertex  $x_{i+1}^{(0)}$  can be tested against the spread- $k$  condition. Also, the equivalent changes and equivalent vertices in the permuted sequences can be deduced and tested. Therefore, the backtracking search is subject to many constraints. If an initial sequence is found that reaches  $x_0^{(1)}$ , then each permuted sequence also reaches the start of the next sequence, and this finally defines  $L$  and the whole coil of  $N$  vertices, with  $N = LP$ .

To summarise this algorithm: an initial leap defines the leap from the starting-point of the initial sequence,  $x_0^{(0)}$ , to the starting-point of the first permuted sequence,  $x_0^{(1)}$ . A permutation defines how the initial leap changes into subsequent leaps. The algorithm looks for successful combinations of initial leap and permutation, which define a skeleton of  $P$  starting-points that lead back to the initial vertex. Then a search is made for a

transition sequence from  $x_0^{(0)}$  to  $x_0^{(1)}$ . During this search, each proposed transition defines a new change in each permuted sequence. There are two main advantages of this algorithm over a simple backtracking search: permuted copies of every proposed transition are additional constraints; and the initial sequence can be short but still produce a long coil.

Another example is supplied, again in  $d = 6$ . Consider a permutation (345201) and an initial leap of (0,1,1,0,0,0). Five permuted leaps are then (0,0,0,0,1,1), (1,1,0,0,0,0), (0,0,0,1,1,0), (1,0,1,0,0,0) and (0,0,0,1,0,1). It can be seen that the effect of these six leaps is to return to the original vertex, so this defines a skeleton with  $P = 6$ . It turns out that a transition sequence (0,1,2,0), which accomplishes the initial leap, is compatible with the permuted sequences (3,4,5,3), (2,0,1,2) and so on that accomplish the permuted leaps, and a permuted coil of  $N = 24$  is formed: (0,1,2,0,3,4,5,3,2,0,1,2,5,3,4,5,1,2,0,1,4,5,3,4).

In the first step, permutations of  $\{0, 1, \dots, d - 1\}$  are considered conjugate if they have the same set of cycle lengths. Therefore, a method of generating non-conjugate permutations is to consider all partitions of the integer  $d$ . For each partition, a cycle is generated with that length. To generate the partitions, Algorithm 7.2.1.4P in [10] would be suitable. This generates partitions in reverse lexicographic order, from ‘ $d$ ’ to ‘11...1’. It was sometimes found to be efficient to concentrate on permutations with long cycles, when an exhaustive search was prohibitive. Therefore a variant of this algorithm was developed, to generate partitions in lexicographic order.

### III. EXAMPLES OF PERMUTED CIRCUIT CODES

For  $d = 10$  and 11, the construction algorithm in the previous section has been used to produce  $d$ -coils

of lengths 348 and 640, longer than the previously known longest, 344 and 630 [4]. For  $d$ -dimensional circuit codes of spread 3, with  $d = 10$  and 11, the construction produces lengths of 100 and 160, longer than the previous known longest, 86 [5] and 154 [6]. These new circuit codes are detailed in the Appendix.

For all  $d \geq 2$ , the transition sequence  $(0, 1, \dots, d - 1, 0, 1, \dots, d - 1)$  defines a  $d$ -dimensional circuit code of length  $2d$  and spread  $d$ . This can be described as a permuted circuit code with a length-1 initial sequence,  $\mathbf{c}^{(0)} = (0)$ , with  $\pi : i \mapsto (i + 1) \bmod d$ .

For all  $d \geq 3$ , a permuted  $d$ -coil  $C$  of length  $2d$  is defined by  $\mathbf{c}^{(0)} = (1, 0)$  and  $\pi : i \mapsto (i + 1) \bmod d$ . This coil contains all  $d$  neighbors of a vertex. (The vertex itself is not part of the coil, of course.) This is most easily seen by taking the first vertex of the circuit to be  $\{1, 0, 0, \dots, 0\}$ , with a single 1-coordinate in coordinate 0. Each pair of changes:

1 0

2 1

3 2

etc.

adds a new 1-coordinate and cancels the previous one, so all vertices with a single 1-coordinate are in the coil  $C$ . These are precisely the  $d$  neighbors of  $\{0, \dots, 0\}$ . Any vertex linked in  $C$  to one of these neighbors has two 1-coordinates, and must therefore be linked to the neighbor with the other 1-coordinate to preserve the spread of 2. Thus this is the unique  $d$ -coil, up to isomorphism in  $I^d$ , where a vertex not in the coil has all  $d$  neighbors in  $C$ . For  $d = 3$ , this  $d$ -coil is identical to the one in the previous paragraph.

A special case of a permuted circuit code is where the initial sequence is repeated once, unchanged:  $(c_0, \dots, c_{L-1}, c_0, \dots, c_{L-1})$ . With this property, a  $d$ -coil

may be called ‘natural’ [7] or ‘symmetric’ [8]. These coils can be regarded as permuted circuit codes with period 2 and the identity permutation. An exhaustive search has been made for 8-coils of this type, again taking the approach of proposing trial vectors for the initial leap to  $x_L$ . This puts additional constraints on the search that starts at  $x_0$ . The longest results have length 94 (compared to 96 for the longest known general 8-coil [5]); an example is given in the Appendix.

The definition of a permuted circuit code can be extended to allow the final permuted sequence to be truncated; the transition sequence is then

$$(\mathbf{c}^{(0)}, \mathbf{c}^{(1)}, \dots, \mathbf{c}^{(P-2)}, c_0^{(P-1)}, c_1^{(P-1)}, \dots, c_i^{(P-1)})$$

with  $i < L - 1$ . An example is  $d = 5$ ,  $\mathbf{c}^{(0)} = (1, 0, 3)$  and  $\pi : i \mapsto (i + 1) \bmod d$ , which ends with  $N = 10$ . When the algorithm was modified to find examples of these special cases, the results were generally shorter than those from full permuted repetitions with similar leap periods.

#### IV. COMPUTATION TIMES

Given a permutation and an initial leap, the search for an initial sequence may be highly constrained, because every change  $c_i^{(0)}$  defines other changes  $c_i^{(1)}$  etc., and all the new occupied vertices must avoid all other vertices in the coil, with spread  $k$ . Also, if  $P$  is large, then only a short initial sequence is needed to produce a long coil. These considerations can make the searches relatively quick.

Example computation times are stated as CPU time for a single processor on an Intel Q8200 Core2 Quad 2.33GHz processor, running the gcc 3.4.5 compiler in Microsoft Vista.

An exhaustive search for permuted 9-coils with leap periods  $P \geq 12$  took 1 minute; the longest result has

length 180 with  $P = 12$ . An exhaustive search for permuted 10-coils with leap periods  $P > 12$  took 80 minutes; the longest result has length 320 with  $P = 16$ . The successful search with  $P = 12$  took 4120 minutes (2.8 days). An exhaustive search for permuted 11-coils with leap periods  $P > 22$  took 1320 minutes (0.9 days); the longest result has length 576 with  $P = 24$ . An exhaustive search for 11-coils with  $P = 22$  took 6.9 weeks, but produced its first length-638 result after approximately 2 days. The length-640 11-coil mentioned in Section II was found by a restricted search of  $P = 20$ .

The computational times taken by searches for initial sequences can be compared to the time taken by an exhaustive search for the longest 7-coils, resulting in length 48, using the method of Section VII. This search took 3480 minutes (2.4 days) on the computer mentioned above. This time is quoted because of the difficulty of like-for-like comparisons with the time taken by the first reported exhaustive search [2]. An exhaustive search for the longest 8-coil is clearly prohibitive using current methods.

The search for permuted  $d$ -coils becomes increasingly difficult for  $d$  larger than 12: large leap periods are relatively quickly found to be unsuccessful, and short leap periods take prohibitively long times to search exhaustively. This construction is not expected to have a large useful range of  $d$  without modification.

#### V. USE OF GRAPHS TO COMPARE AND INVESTIGATE SEQUENCES OF HYPERCUBE VERTICES

In Section II, it was efficient to test the skeleton of known vertices  $(x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(P-1)}, x_0^{(0)})$ , which represented the starts of all the initial and permuted sequences. The test was whether this skeleton was isomorphic with any previously considered skeleton.

In general, it is useful to be able to compare sequences

of vertices. For example, in searching for coils, it is efficient to reject initial sequences that are isomorphic to others that have already been tried. In the current work, comparisons such as these have been implemented by representing the vertex sequences in a graph related to the hypercube.

When the hypercube is regarded as a graph, the relationship between the vertices are defined by connecting each vertex to its neighbours. Therefore, a sequence of vertices cannot generally then be defined by simple connections in the resulting graph, because all permissible connections have already been made. Therefore, an extended graph is used here, where the vertices representing those of the hypercube are not connected to each other, except when they are linked in a sequence. Additional coordinate vertices are used to define the relationship between the original vertices. The  $d$ -dimensional extended graph contains  $2^d$  original vertices and  $2d$  coordinate vertices. The coordinate vertices come in pairs, each pair representing the 0- and 1-coordinates in one of the  $d$  dimensions. Each original vertex is connected to  $d$  coordinate vertices, one from each pair, with the choice of 0 or 1 defined by the relevant coordinate of the corresponding vertex in the hypercube. The vertices in each coordinate pair are connected to each other in the extended graph. The extended graph for  $d = 3$  is shown in Figure 1.

Every automorphism of the hypercube has an equivalent automorphism in the equivalent extended graph, with original vertices mapping only onto original vertices, and coordinate vertices onto coordinate vertices. For example, a reflection in one coordinate of the hypercube is equivalent to swapping the corresponding pair of coordinate vertices. A sequence of vertices in the hypercube can be represented in the extended graph by connecting original vertices; sequences that are equivalent

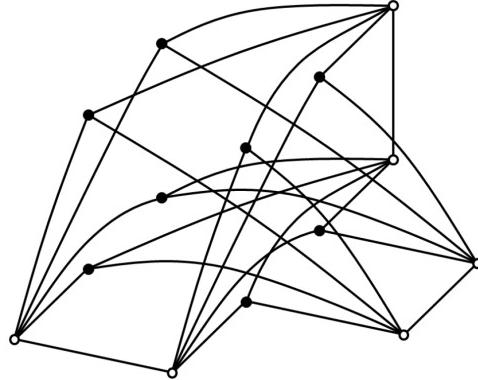


Fig. 1. The extended graph representing the cube  $I^3$ . Filled circles represent original vertices (in the positions of a plane projection of a geometric cube); open circles represent coordinate vertices.

(up to symmetries of the hypercube) are isomorphic in the extended graph. Instances of the extended graph with connected sequences can be compared efficiently in programs using, for example, the NAUTY software [9]: two extended graphs are built up, and NAUTY then converts them into canonical forms. If and only if these canonical forms are identical, then the two vertex sequences in the hypercube are equivalent.

There are ‘brute-force’ approaches for comparing vertex sequences in the hypercube. For example, simple paths can be compared by renumbering the coordinates so that the transition sequence has its lowest possible lexicographical order, if necessary evaluating the forwards and backwards versions and selecting the lower of the two. Simple circuits can be compared by similarly renumbering the coordinates for each starting point, in both directions, and selecting the lowest of all these renumbered sequences. The extended graph method is generally slower than these brute-force approaches for simple paths and circuits, but it is more flexible: for example, it can be applied to chains or skeletons of vertices that are not connected in the hypercube.

The extended graph can also be used to study sym-

metries of paths or circuits; graph analysis software such as NAUTY can report the orbits of the vertices of an extended graph with connected original vertices. While the term ‘symmetric’ has sometimes been used for period-2 permuted  $d$ -coils with the identity permutation (as mentioned above), this is not recommended, because there are many other possible symmetries. For example, any permuted sequence will have order- $P$  symmetry. The description *natural* is therefore preferable, although it is not entirely clear how it was derived.

As an example of symmetry in hypercube paths, a  $d$ -coil can be defined as *invertible* if the two oriented circuits are isomorphic to each other. In the isomorphism, one pair of vertices or changes must remain fixed; we may therefore distinguish between *vertex-fixed* or *change-fixed* inversions. In different isomorphisms, a coil might have both types; an example is the highly symmetrical coil  $(0, 1, \dots, d-1, 0, 1, \dots, d-1)$ , but no other examples are known.

Examples of invertible coils are the three longest 5-coils, with  $N = 14$ : two are vertex-fixed invertible and the other is change-fixed invertible. One vertex-fixed invertible coil is shown below and repeated in reverse order:

0123142 1023124

4213201 2413210.

It can be seen that the two presentations are isomorphic via a permutation of coordinates  $(04)(12)3$ , so the initial vertex is fixed by this inversion. The same would not be true if the transition sequence were cycled to start at any other vertex except the ‘opposite’ vertex (at the position of the spaces in the lines above). So, for example, there is no permutation of coordinates that maps the transition sequence onto its inverse if the initial vertex is moved by one place:

12314210231240

04213201241321.

The other vertex-fixed invertible 5-coil with  $N = 14$  is natural:

0123042 0123042

2403210 2403210.

The change-fixed invertible 5-coil with  $N = 14$  is shown below and repeated in reverse order, while fixing the two changes:

0 120324 0 123024

0 420321 0 423021.

Using the extended graph technique, we find that all  $d$ -coils with  $d \leq 5$  are invertible, but longer invertible  $d$ -coils appear to be rare. Of the 4 maximal 6-coils ( $N = 26$ ), only 1 is invertible. The usefulness of the extended graph technique can be illustrated by asking questions: which of the following two 6-coils is invertible, and is it vertex- or change-fixed?

01203143052351035230420135

01203104201350120310421035.

Of the 758 maximal 7-coils ( $N = 48$ ), only 37 are invertible; all except one of these have vertex-fixed inversions. None of the longer  $d$ -coils mentioned in this paper are invertible.

## VI. CONSTRUCTION USING LOWER-DIMENSIONAL SNAKES

An unrelated construction is briefly mentioned here. If the transition sequences **b** and **c** define two  $d$ -snakes, then the transition sequence  $(\mathbf{b}, d, \mathbf{c})$  may define a  $(d+1)$ -snake, or  $(\mathbf{b}, d, \mathbf{c}, d)$  may define a  $(d+1)$ -coil.

Given two  $d$ -snakes, it will generally be necessary to permute one of the change sequences to search for a successful combination. An efficient way to do this is to add vertices of the permuted second snake to the combined sequence one by one, generating each element of the permutation only when required, and proceeding

only when a suitable element can be found. Algorithm 7.2.1.2X in [10] can be used, because it can be used to generate incomplete permutations exactly as required. The steps in the method are therefore as follows:

- Start with a snake consisting of the first sequence.
- Consider each change in the second sequence, in order. If it is a change number that has not yet been assigned a permutation, generate a new permutation of that change number.
- Append the permutation of each change to the current snake.
- If the new snake disobeys the required spread condition, backtrack to the most recently generated permutation and generate a new permutation. If there are no more possibilities, remove the assigned permutation and backtrack to the next most recent permutation, and so on.

During the exhaustive search for 7-coils, 7-snakes were recorded. Pairs of these were combined into 8-snakes, which were then similarly combined. For  $d = 9$ , this resulted in a coil of length 188 and a snake of length 190, longer than the previously known longest, 180 and 188 [11]. The computation times for attempting to combine 9-snakes are not prohibitive, but there is a very large number of candidates of suitable lengths to form long 10-coils. This construction is not expected to be widely useful for large  $d$ .

## VII. RESULTS FROM DIRECT SEARCHES

Backtracking searches were conducted for circuit codes. Partial sequences were rejected if any subsequence, running forwards or backwards, could be renumbered to a lower number than the starting subsequence. This is not an exhaustive search for snakes, but it avoids the wasted effort of finding circuits from multiple

starting points. Every subsequence was tested at every step; more efficient strategies may well be available.

Exhaustive searches confirmed the optimality of known sequences [12]: length 46 for  $d = 10$ , spread 4; and length 40 for  $d = 11$ , spread 5. Additionally, circuit codes of lengths 58, 58, and 50 were found for  $d = 9$ , spread 3, for  $d = 12$ , spread 5, and for  $d = 13$ , spread 6 — longer than the previous known longest, 56 [6], 56 [13], and 48 [12] respectively.

## VIII. CONCLUSIONS

Two methods are presented in this paper for searching for circuit codes. Both methods attempt to reduce the combinatorial explosion of the search by adding constraints. In the first method, permuted sequences are used; the entire sequence is assembled from multiple permuted copies of a shorter sequence. In the second method, the search is effectively for two lower-order sequences that can be combined with only one pair of changes in the highest coordinate. It is remarkable that such constrained searches can be competitive with more general searches, but they have produced new records in several cases that are presented here.

A method is presented for detecting the symmetries of circuit codes through the use of efficient tools for analysing graphs. Circuit codes are defined as circuits on the hypercube graph; the nodes are already linked together before a circuit is specified. Therefore, it is useful to define an extended graph that includes nodes that are closely related to the hypercube's nodes but which are not initially linked. Symmetries equivalent to the hypercube's symmetries are brought about through additional vertices. Examples of the symmetries that can be detected using this method are two different inversions.

APPENDIX  
DETAILS OF CIRCUIT CODES

Some transition sequences are given in full; others can be deduced by permuting the initial sequences.

- Coil, spread 2,  $d = 10$ : initial sequence 01897208469847685740278968076,  $L = 29$ , permutation (123450)(786)9,  $P = 12$ ,  $N = 348$ . The full sequence:

01897208469847685740278968076  
 12698316579658760851386976187  
 23796427089706871602467987268  
 34897538169817682713578968376  
 45698046279628763824086976487  
 50796157389736874635167987568  
 01897208469847685740278968076  
 12698316579658760851386976187  
 23796427089706871602467987268  
 34897538169817682713578968376  
 45698046279628763824086976487  
 50796157389736874635167987568.

- Coil, spread 2,  $d = 11$ : initial sequence 0168763891305943A671 35127A237,  $L = 29$ , permutation (123456789A0),  $P = 22$ ,  $N = 638$ .

- Coil, spread 2,  $d = 11$ : initial sequence A04A82A73A 26A38A27A4 8162A648A4 02,  $L = 32$ , permutation (1234567890)A,  $P = 20$ ,  $N = 640$ . In this coil, it is noteworthy that more than one quarter of the changes are in a single coordinate. The full sequence:

A04A82A73A26A38A27A48162A648A402  
 A15A93A84A37A49A38A59273A759A513  
 A26A04A95A48A50A49A60384A860A624  
 A37A15A06A59A61A50A71495A971A735  
 A48A26A17A60A72A61A82506A082A846  
 A59A37A28A71A83A72A93617A193A957

A60A48A39A82A94A83A04728A204A068  
 A71A59A40A93A05A94A15839A315A179  
 A82A60A51A04A16A05A26940A426A280  
 A93A71A62A15A27A16A37051A537A391  
 A04A82A73A26A38A27A48162A648A402  
 A15A93A84A37A49A38A59273A759A513  
 A26A04A95A48A50A49A60384A860A624  
 A37A15A06A59A61A50A71495A971A735  
 A48A26A17A60A72A61A82506A082A846  
 A59A37A28A71A83A72A93617A193A957  
 A60A48A39A82A94A83A04728A204A068  
 A71A59A40A93A05A94A15839A315A179  
 A82A60A51A04A16A05A26940A426A280  
 A93A71A62A15A27A16A37051A537A391.  

- Coil, spread 3,  $d = 10$ : initial sequence 26014,  $L = 5$ , permutation (1234567890),  $P = 20$ ,  $N = 100$ .
- Coil, spread 3,  $d = 11$ : initial sequence 0A184A5234,  $L = 10$ , permutation (12345670)(98)A,  $P = 16$ ,  $N = 160$ .
- Coil, spread 3,  $d = 11$ : initial sequence 0623184A,  $L = 8$ , permutation (1234567890)A,  $P = 20$ ,  $N = 160$ .
- Natural coil,  $d = 8$ : transition sequence 0314035046 0340745135 6253157407 5305670517 0317436 twice,  $N = 94$ .
- Coil,  $d = 9$ , from construction in Section VI: transition sequence 0123043254 2134256352 1324532105 1245231524 6142315712 3152413210 4213245321 3461235421 3253045213 2458032105 1245231524 6142312541 2304325421 3425635213 4732134253 1230523125 4123156321 4523124105 42312548,  $N = 188$ .
- Snake,  $d = 9$ , from construction in Section VI: transition sequence 0120314021 0541021432 1026431450 4134210431 4501432731 2014301263 2143053102 3053145036 0431402143 1046806104

- 3145014310 6302143203 5043203145 3654031405  
 4375314021 4310451341 0214316504 5314504120  
 4501430540,  $N = 190$ .
- Coil, spread 3,  $d = 9$ : 0123041502 1603570132  
 4038175014 5671536012 3674563017 60581735,  
 $N = 58$ .
  - Coil, spread 5,  $d = 12$ : 0123450617 2803196A04  
 72160548B7 014A836105 82A9167854  
 0613A84B,  $N = 58$ .
  - Coil, spread 6,  $d = 13$ : 0123456071 82930A142B  
 9C630529A7 60124A8305 629B4C5A,  $N = 50$ .

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#### REFERENCES

- [1] N. J. A. Sloane, "The on-line encyclopedia of integer sequences," published electronically at [www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/), Sequence A000937, accessed 8 March 2010, 2008.
- [2] K. J. Kochut, "Snake-in-the-box codes for dimension 7," *J. Combin. Math. Combin. Comput.*, vol. 20, pp. 175 – 185, 1996.
- [3] V. Klee, "What is the maximum length of a d-dimensional snake?" *Amer. Math. Monthly*, vol. 77, pp. 63 – 65, 1970.
- [4] D. Casella and W. Potter, "Using evolutionary techniques to hunt for snakes and coils," in *Proceedings of the 2005 IEEE Congress on Evolutionary Computing*, Edinburgh, Scotland, September 2005.
- [5] K. Paterson and J. Tuliani, "Some new circuit codes," in *Proceedings of the 1997 IEEE International Symposium on Information Theory*, Ulm, Germany, June 1997.
- [6] I. Zinovik, D. Kroening, and Y. Chebiryak, "Computing binary combinatorial Gray codes via exhaustive search with SAT solvers," *IEEE Trans. Inform. Theory*, vol. 54, pp. 1819 – 1823, 2008.
- [7] W. Kautz, "Unit-distance error-checking codes," *IRE Trans. Electronic Computers*, vol. 7, pp. 179 – 180, 1958.
- [8] R. C. Singleton, "Generalized snake-in-the-box codes," *IEEE Trans. Electron. Comput.*, vol. 15, pp. 596 – 602, 1966.
- [9] B. D. McKay, "Practical graph isomorphism," *Congressus Numerantium*, vol. 30, pp. 45 – 87, 1981, NAUTY Version 2.4 from <http://cs.anu.edu.au/~bdm/nauty/>, accessed 8 March 2010.
- [10] D. E. Knuth, *The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 1.* Addison-Wesley, 2011.
- [11] D. R. Tuohy, W. D. Potter, and D. A. Casella, "Searching for snake-in-the-box codes with evolved pruning models," in *Proceedings of the 2007 International Conference on Genetic and Evolutionary Methods, GEM2007*, Las Vegas, USA, June 2007.
- [12] K. Paterson and J. Tuliani, "Some new circuit codes," *IEEE Trans. Inform. Theory*, vol. 44, pp. 1305 – 1309, 1998.
- [13] Y. Chebiryak and D. Kroening, "An efficient SAT encoding of circuit codes," in *Proceedings of the 2008 IEEE International Symposium on Information Theory and its Applications*, Auckland, New Zealand, December 2008.